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THE PLANE REMOTE TURBULENT WAKE IN THE LIGHT OF THE GENERALIZED KÁRMÁN THEORY

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The formula for the tangential Reynolds stresses proposed for the plane turbulent boundary flows [1, 2]

$$\tau^\vee = \mu \varkappa_n T^{\vee n} \frac{\partial u^\vee}{\partial y^\vee}, \quad T^\vee = \left|\frac{\partial u^\vee}{\partial y^\vee}\right|^3 \left[\nu \left|\frac{\partial^2 u^\vee}{\partial y^{\vee 2}}\right|^2\right]^{-1} \tag{0.1}$$

is extended to embrace the problem of a plane turbulent wake in the region in which it is self-similar (without altering the experimental constants n and $\varkappa_n$). It is shown that this yields results similar to those given by the Schlichting theory, thus providing a link between the empirical constants of the latter theory and the constants of the boundary layer turbulence. In particular, this made possible the computation of the value of the constant $\varkappa$ appearing in the universal logarithmic law, using only the experiments dealing with the turbulent wakes. It was found that the value of $\beta = 0.18$ recommended by Schlichting has the corresponding value of $\varkappa = 0.42$, which deviates only by 5% from the generally accepted value obtained from the Nikuradse experiments.

When $n = 1$ and $\varkappa_n = 0.16$ (0.1) becomes the known Kármán formula. So far, the latter formula has not been used in the problems of free turbulence, since the stream and wake velocity profiles have points of inflection (at these points $\partial^2 u / \partial y^2 = 0$ and hence by virtue of (0.1), $|\tau| \to \infty$ which is absurd). Nevertheless, (0.1) can be applied to streams and wakes provided that we assume that the derivative $\partial^2 u / \partial y^2$ does not vanish at the point of inflection of the velocity profile, but passes through it undergoing a finite discontinuity of the type $\pm a$. The discussion which follows will show that such an assumption produces results which are sufficiently near to the experimental data (while retaining the values of n and $\varkappa_n$ which were recommended [1, 2] for the problems of the boundary layer turbulence).

1. Using the impulse loss width b_{**} as the characteristic dimension, we pass to the dimensionless variables (here the ticks $^\vee$ denote the corresponding dimensional quantities)

$$x = \frac{x^\vee}{b_{**}} \frac{1}{R_{**}^{1-n}}, \quad R_{**} = \frac{U_\infty b_{**}}{\nu}$$

$$y = \frac{y^\vee}{b_{**}}, \quad b = \frac{b^\vee}{b_{**}}$$

$$u = \frac{U_\infty - u^\vee}{U_\infty}, \quad v = \frac{v^\vee}{U_\infty} R_{**}^{1-n}$$

where $2b^{\vee}(x^{\vee})$ denotes the width of the wake, which is finite according to both the present theory and the first Prandtl theory.

Applying the usual simplifications (see [3]) and utilizing the formula (0.1), we reduce the problem of the plane remote turbulent wake to the following:

$$\varkappa_n \frac{\partial}{\partial y}\left(T^n \frac{\partial u}{\partial y}\right) = -\frac{\partial u}{\partial x}, \quad T = \left|\frac{\partial u}{\partial y}\right|^3 \left|\frac{\partial^2 u}{\partial y^2}\right|^{-2} \tag{1.1}$$

$$\tau = 0, \quad y = 0; \quad u = 0, \ y = b$$

$$(\tau = \varkappa_n T^n \partial u / \partial y)$$

$$\int_0^b u \, dy = 1$$

The above formulas must be supplemented by the equation of continuity

$$\partial u / \partial x - \partial v / \partial y = 0$$

which is necessary for the determination of the velocity v.

We seek the solution of the problem in question in the self-similar form

$$\psi = x^{\alpha} F(\eta), \quad \eta = y / x^{\beta} \quad (u = \partial\psi / \partial y, \ v = \partial\psi / \partial x) \tag{1.2}$$

where ψ is the dimensionless stream function, while α and β are constants determined in the course of solution, from the conditions of its self-similarity.

Substituting (1.2) into (1.1) we obtain

$$\alpha = 0, \quad \beta = {}^1/_2 \tag{1.3}$$

and (1.1) becomes an ordinary differential equation (in what follows, a prime denotes a derivative with respect to η)

$$\varkappa_n (t^n F'')' = -{}^1/_2 (F' + \eta F''), \quad t = |F''|^3 \, |F'''|^{-2} \tag{1.4}$$

This can be integrated once, to assume the following form:

$$\varkappa_n t^n F'' = -{}^1/_2 \eta F' \tag{1.5}$$

where it is already assumed that $\tau = 0$ when $\eta = 0$. The corresponding boundary conditions are

$$\begin{aligned} &F = 0, \quad \eta = 0 \\ &F = 1, \quad F' = 0, \quad \eta = \eta_0 \end{aligned} \tag{1.6}$$

where η_0 denotes the coefficient appearing in the equation for the wake boundary

$$b = \eta_0 x^{1/2} \tag{1.7}$$

The condition that $F = 1$ when $\eta = \eta_0$ emerges from the integral condition (1.1) after substituting into it u and y in accordance with (1.2) and (1.3).

2. Although the number of the boundary conditions (1.6) is equal to the order of the differential equation (1.5), nevertheless the problem still remains underdefined since the parameter η_0 is not known. The necessity clause in the auxilliary condition is due to the fact that when formula (0.1) is used, the order of the differential equation is higher by one, than that encountered in the traditional approach to the problems of free turbulence. In the problems of the boundary layer turbulence such an auxilliary condition is supplied by the Kármán's assumption that $\partial u / \partial y \to \infty$ at

the wall, which was utilized in [1, 2].

In computing the turbulent flows and wakes we use, as the axilliary condition, the natural assumption that the inflection points on the velocity and wake profiles coincide with the position of maximum of the tangential stress. Using (1.4), we can write this condition as follows:

$$F' = -\eta F'', \quad \eta = \eta_1 \quad (b_1 = b_1^{\vee} \,/\, b_{**} = \eta_1 x^{1/2}) \tag{2.1}$$

where $y^{\vee} = b_1^{\vee}(x^{\vee})$ is the coordinate of the point of inflection.

An unknown parameter η_1 appears in (2.1). Using the available experimental data on the profiles of plane turbulent wakes and free streams, we assume that the point of inflection of the velocity profile is situated at the distance $0.4b^{\vee}$ from its axis. The expression corresponding to this assumption is

$$\eta_1 = 0.4\eta_0 \tag{2.2}$$

Adding the equations (2.1) and (2.2) to the boundary conditions (1.6) formulated earlier, makes the problem fully defined.

3. As we already said, the solution of the problem in question must be sought in the class of functions $F(\eta)$, with the derivative F''' undergoing a finite discontinuity of the form $\pm a$ at the point $\eta = \eta_1$ (this corresponds to a finite discontinuity in the derivative $\partial^2 u \,/\, \partial y^2$ at the point of inflection of the velocity profile). From this we have

$$F(\eta) = \begin{cases} F_1(\eta), & 0 \leqslant \eta \leqslant \eta_1 \\ F_2(\eta), & \eta_1 \leqslant \eta \leqslant \eta_0 \end{cases} \tag{3.1}$$

where $F_1(\eta)$, $F_2(\eta)$ are analytic functions satisfying the differential equations following from (1.5), (1.6), with the boundary conditions

$$\varkappa_n t_1^n F_1'' = -{}^1/_2 \eta F_1', \quad 0 \leqslant \eta \leqslant \eta_1 \tag{3.2}$$

$$\varkappa_n t_2^n F_2'' = -{}^1/_2 \eta F_2'', \quad \eta_1 \leqslant \eta \leqslant \eta_0$$

$$F_1 = 0, \quad \eta = 0; \quad F_2 = 1, \ F_2' = 0, \quad \eta = \eta_0 \tag{3.3}$$

and the "matching" requirement

$$F_1 = F_2, \quad F_1' = F_2', \quad F_1'' = F_2'', \quad F_1''' = -F_2''', \quad \eta = \eta_1 \tag{3.4}$$

Equations (2.1) and (2.2) define the position of the matching point.

When integrating the equations (3.2) numerically on a digital computer, it pays to remember that their solutions have the following properties:

$$F_1 \geqslant 0, \quad F_1' > 0, \quad F_1'' < 0, \ F_1''' < 0$$

$$F_2 > 0, \quad F_2' > 0, \quad F_2'' < 0, \quad F_2''' > 0$$

We note that neither F_1'' is zero when $\eta = 0$ nor F_2'' is zero, when $\eta = \eta_0$, i.e. the velocity profile in the wake has a sharpness, according to the theory in question, on the axis of the flow, as well as at its boundaries. The proposed theory resembles in this respect the first Prandtl theory. On the other hand, the boundary conditions $\tau = 0$ when $\eta = 0$ and $\eta = \eta_0$ hold, because F_1''' and F_2''' tend, at the points indicated, to $\mp\infty$ respectively, in accordance with the following asymtotic laws:

$$F_1''' \div -\frac{1}{\eta^{1/2n}}, \quad F_2''' \div \frac{1}{(\eta_0 - \eta)^{1/2n}}$$

where $\div$ denotes the proportionality.

4. The problem formulated above was solved on a digital computer for the following two sets of values:

$$n = {}^2/_3, \qquad \varkappa_n = 0.55$$
$$n = {}^4/_5, \qquad \varkappa_n = 0.59$$

which were arrived at [1, 2] using the experimental data obtained by Nikuradse and others while investigating turbulent flows in pipes. The computations have shown that the values of $F(\eta)$ and its first three derivatives coincide in both cases to within the first two significant figures, and this makes it possible to combine both sets of calculations in a single Table 1.

5. The problem of remote turbulent wake was solved (using the first Prandtl theory) by Schlichting (see [3]). The solution yields the following formulas:

$$u_m = \frac{U_\infty - u_m^\vee}{U_\infty} = \frac{1.90}{\bar{x}^{1/2}} \tag{5.1}$$

$$\frac{\tau_M^\vee}{\rho u_m^{\vee 2}} = 0.065, \quad b = 1.14\bar{x}^{1/2}$$

$$\bar{x} = xR_{**}^{1-n} = \frac{x^\vee}{b_{**}} \frac{4x^\vee}{c_w d}$$

where $U_\infty - u_m^\vee$ denotes the velocity drop at the wake axis and $\tau_M^\vee$ is the maximum value of the tangential Reynolds stress.

The corresponding formulas obtained from the proposed novel solution, have the form

$$u = \frac{F'(0)}{x^{1/2}} = \frac{0.79R_{**}^{(1-n)/2}}{\bar{x}^{1/2}} \tag{5.2}$$

$$\frac{\tau_M^\vee}{\rho u_m^{\vee 2}} = \frac{1}{2} \frac{\eta_1 F'(\eta_1)}{[F'(0)]^2 R_{**}^{1-n}} = \frac{0.388}{R_{**}^{1-n}}$$

$$b = \eta_0 x^{1/2} = 2.69R_{**}^{-(1-n)/2}\bar{x}^{1/2}$$

The numerical coefficients appearing in (5.2) were taken from Table 1.

Substituting in (5.2) $n = {}^2/_3$, $R_{**} = 194$ or $n = {}^4/_5$, $R_{**} = 6475$, yields identical results, namely

$$u_m = \frac{1.90}{\bar{x}^{1/2}}, \quad \frac{\tau_m^\vee}{\rho u_m^{\vee 2}} = 0.067, \quad b = 1.12\bar{x}^{1/2} \tag{5.3}$$

The above formulas practically coincide with the Schlichting's solution (5.1). Using the experimental data on the hydrodynamic resistance of circular cylinders within the range $\mathrm{Re} = U_\infty d / \nu = 10^4 \div 10^5$ of Reynolds numbers, we find that $\mathrm{Re} = (4 \div 5)R_{**}$. It follows therefore that e.g. the value of $\mathrm{Re} = 1000$ corresponds to $R_{**} = 194$

and $\mathrm{Re} = 3 \cdot 10^4$ to $R_{**} = 6475$.

All this leads to conclusion (remembering that the Schlichting theory was matched with the experimental data by choosing the value of the parameter $\beta = 0.18$) that the proposed theory should be, in the case $n = {}^4/_5$, in reasonable agreement with the experiment within the range of Reynolds numbers $\mathrm{Re} = 10^4 \div 10^5$. However, the proposed theory, although coinciding with the Schlichting's theory at $\mathrm{Re} = 3 \cdot 10^4$, should give somewhat different results at other values of the Reynolds numbers, predicting a reduction in the thickness of the wake and an increase in the value of u_m proportional to $\mathrm{Re}^{0.1}$ (compared with the formulas (5.1)).

6. In considering the boundary layer flows (in the framework of the proposed theory based on the assumption that the thickness of the boundary layer is neglected) we have excluded the Kármán case of $n = 1$, since it would not allow the condition $u = 0$ at the wall to hold. In the case of free turbulent flow, the value $n = 1$ becomes admissible, Equations (3.2) were integrated numerically on a digital computer also for this case, and the constant $\varkappa_n$ (which Kármán assumed, on the strength of the Nikuradse experiments with pipes, to be equal to 0.16) was not set in advance, but was determined in the course of solution from the requirement that

$$\eta_0 = 2\sqrt{10}\,\beta = 1.14 \quad (\beta = 0.18) \tag{6.1}$$

The last equation guarantees that the width of the wake is determined (in computing it according to the Kármán's variant $n = 1$) using the formula employed in the Schlichting's method, i.e. the penultimate formula of (5.1). Solving the equations (3.2) for $n = 1$ we established, as a result, that the conditions (3.3), (3.4), (2.1), (2.2) and in addition (6.1), all hold, provided that $\varkappa_n = \varkappa^2 = 0.176$, and this yields $\varkappa = 0.42$.

This result merits special attention. Indeed, it implies that the empirical constant $\varkappa$ appearing in the universal logarithmic law, the determination of which was based, up to now, on the experiments concerning flows in pipes and in boundary layers, can be determined (with an error of only 5%) from the measurements showing the change in the width of the turbulent wake, with which it has apparently no connection whatsoever. It is obvious now that β and $\varkappa$ are interrelated, and knowing one we can obtain the other. This, together with the results obtained in Sect. 5 for the values of $n \neq 1$, illustrate the universal character of (0.1), which embrace the problems of the boundary layer turbulence and of free turbulence, while retaining the same empirical constants for both classes of problems. Table 2 gives the values of the function $F(\eta)$ and its first three derivatives obtained from (3.2) for $n = 1, \varkappa_n = 0.176$. The values in brackets appearing in the column for $F'(\eta)$ correspond to the Schlichting solution

$$u = 1.90\left[1 - \left(\frac{y}{1.14}\right)^{3/2}\right]^2 \bar{x}^{-1/2} \approx F'(\eta)\,\bar{x}^{-1/2} \tag{6.2}$$

In Fig. 1 the velocity drop profile in the turbulent wake corresponding to the Kármán's case is shown with the solid line, and points depict the velocity profile given by (6.2). Fig. 1 and Table 2 both show that these profiles practically coincide over the whole width of the wake. Similar conclusion is reached by comparing the Schlichting profile (6.2) with the velocity profile given in Table 1 for $n = {}^2/_3$ and $\varkappa_n = {}^4/_5$.

Table 1

η	F	F'	F''	F'''	t
0.00	0.00	0.79	—0.16	—1000	0.00
0.10	0.08	0.77	—0.23	—0.27	0.17
0.20	0.15	0.74	—0.25	—0.20	0.39
0.40	0.30	0.69	—0.29	—0.17	0.81
0.60	0.43	0.63	—0.32	—0.17	1.10
0.80	0.55	0.56	—0.36	—0.20	1.21
1.00	0.65	0.48	—0.40	—0.24	1.15
1.075	0.69	0.45	—0.42	±0.26	1.08
1.20	0.74	0.40	—0.34	+0.22	1.19
1.4	0.81	0.33	—0.35	0.18	1.30
1.60	0.87	0.26	—0.32	0.16	1.31
2.00	0.95	0.15	—0.26	0.13	1.03
2.40	0.99	0.05	—0.21	0.15	0.41
2.69	1.00	0.000	—0.13	2.95	0.00

Table 2

η	F	F'	F''	F'''
0.00	0.000	1.900 (1.900)	—1.000	—1000
0.10	0.186	1.774 (1.802)	—1.404	—2.777
0.20	0.356	1.620 (1.632)	—1.686	—2.964
0.30	0.509	1.435 (1.422)	—2.013	—3.665
0.40	0.642	1.214 (1.190)	—2.443	—5.080
0.456	0.706	1.068	—2.763	∓6.492
0.50	0.750	0.953 (0.958)	—2.503	+5.387
0.60	0.834	0.726 (0.724)	—2.052	+3.785
0.70	0.896	0.538 (0.512)	—1.723	+2.869
0.80	0.942	0.379 (0.322)	—1.466	+2.314
0.90	0.973	0.244 (0.170)	—1.252	+1.986
1.00	0.992	0.128 (0.060)	—1.061	+1.866
1.10	0.999	0.031	—0.862	+2.364
1.14	1.000	0.000	—0.713	+12.26

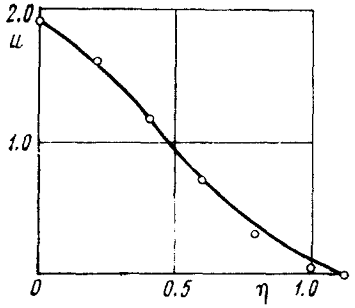

Fig. 1

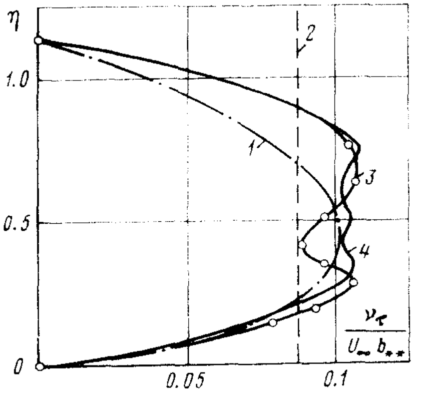

Fig. 2

7. Comparison of the proposed theory with the generally accepted method of computing the turbulent wakes also in the part of the law of variation in the turbulent viscosity ν_τ across the wake, is of interest.

According to the first Prandtl theory [3] we have

$$\frac{\nu_\tau}{U_\infty b_{**}} = 0.216\left(\frac{y^\vee}{b}\right)^{1/2}\left[1-\left(\frac{y^\vee}{b}\right)^{3/2}\right] \tag{7.1}$$

while his second theory [3] gives

$$\nu_\tau / U_\infty b_{**} = 0.0888 \tag{7.2}$$

and the proposed theory yields

$$\frac{\nu_\tau}{U_\infty b_{**}} = \frac{1}{R_{**}^{1-n}}\varkappa_n t^n \tag{7.3}$$

Figure 2 depicts four curves constructed according to the formulas (7.1) (curve *1*), (7.2) (curve *2*) and (7.3) with $R_{**} = 6475$, and the last formula represented by two variants, $n = 1, \varkappa_n = 0.176$ (curve *3*) and $n = 4/5, \varkappa_n = 0.59$ (curve *4*). The value $R_{**} = 6475$ adopted in the last variant corresponds (as was established before) to the case when the values of the width of the wake $\tau_M^\vee$ and $u_m^\vee$ in this variant all coincide with those obtained from the Schlichting formulas.

Figure 2 implies that the proposed theory in both its variants occupies (as regards the variation in the value of ν_τ across the wake) an intermediate position between the first and second theory of Prandtl. In the interval $0.2 < \eta < 0.8$ it yields the values $\nu_\tau / (U_\infty b_{**})$ which differ little from the constant value of 0.11, and thus resemble (7.2). However, at the end points of the interval $0 \leqslant \eta \leqslant \eta_0$ we have, according to the proposed theory, $\nu_\tau = 0$ as in (7.1). The true value lies apparently between the results of the two Prandtl theories. If the first of these theories correctly discerns the decrease in ν_τ to zero at the axis and at the free boundaries of the wake, the second theory correctly indicates that that the curve ν $(y^\vee)$ should rather be box-like than parabolic, exaggerating this by assuming that the relation $\nu_\tau / U_\infty b_{**}$, is independent of $y^\vee$.

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